The induced representation and Frobenius reciprocity

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Abstract

Given a subgroup $H \leq G$, and a representation $W$ of $H$, we may construct a representation of $G$, known as the induced representation and denoted as $\text{Ind} W$. Fulton and Harris give a constructive definition of this object; we shall however give a more theoretical one based on the idea of change the underlying ring of a module. As we shall see, this definition is conceptually clearer and has the Frobenius reciprocity theorem as an easy and transparent corollary.

1 Tensor products over arbitrary rings

Suppose $S$, $R$ and $T$ are (possibly non-commutative) rings. Let $A$ be a left $S$- and a right $R$-module. We denote this by $_S A_R$. In addition, suppose $B$ is a left $R$- and a right $T$-module. As before, we shall denote this as $_R B_T$. We form the tensor product $A \otimes B$ as follows:

1. Take the free vector space $K$ generated by pairs $(a, b)$: $a \in A, b \in B$.

2. Quotient out by the relations:
   
   (a) $(a + c, b) \sim (a, b) + (c, b)$

   (b) $(a, b + d) \sim (a, b) + (a, d)$

   (c) $(ar, b) \sim (a, rb)$

3. Denote the class $[(a, b)]$ as $a \otimes b$

Notice how we used the fact that $A$ is a right $R$-module and $B$ is a left $R$-module in relation 3. The resulting vector space is now a left $S$-module and a right $T$-module: $_S A \otimes _R B_T$. 

1
2 Adjoint Functors

The theory of adjoint functors is too complicated to explain here (that is, I don’t really understand it yet) but I think the simplest and most approachable definition of adjoint functors is this one from Wikipedia which is in terms of hom-sets

**Definition 1.** A hom-set is the set of all morphisms between two objects in a category $C$:

$$\text{hom}(a,b) = \{ f : f \text{ in } C, f : a \to b \} \quad (2.1)$$

hom-sets can carry extra structure. For example $\text{hom}(a,a)$ is always a monoid, for any $a$ in any category $C$. Also, if $C$ is an Abelian Category, then $\text{hom}(a,b)$ is always an abelian group (although I’m still not entirely sure how the binary operation is defined. We can put a subscript: $\text{hom}_C(a,b)$ when we wish to emphasise that $a$, $b$ and all the morphisms in $\text{hom}(a,b)$ live in the category $C$.

**Definition 2.** Given two functors $F : C \to D$ and $G : D \to C$, suppose that for each $c \in C$, $d \in D$ we have an isomorphism $\Phi_{cd}$:

$$\Phi_{cd} : \text{hom}_D(Fc,d) \to \text{hom}_C(c,Gd) \quad (2.2)$$

$F$ and $G$ are adjoint functors if, for every $c \in C$, $d \in D$ such a $\Phi_{cd}$ exists and it satisfies certain naturality conditions (which I don’t fully understand).

The important part is this: let $R\text{mod}$ denote the category of left $R$ modules and $S\text{mod}$ denote the category of left $S$ modules ($R$ and $S$ are (possibly noncommutative) rings). Using notation as in section 1, if we have a bimodule $RX_S$ then we can define a functor:

$$F : S\text{mod} \to R\text{mod} \quad (2.3)$$

$$S Y \mapsto X \otimes Y \quad (2.4)$$

$$S U \mapsto (\text{hom}_R(X,Y)) \quad (2.5)$$

If we recall that $\text{hom}_R(RX_S,RZ)$ (this is a slight abuse of notation. We should write $\text{hom}_{R\text{mod}}(RX_S,RZ)$ but this becomes cumbersome) is a left $S$ module we can define a second functor, $G$:

$$G : R\text{mod} \to S\text{mod} \quad (2.6)$$

$$R U \mapsto (\text{hom}_R(X,Y)) \quad (2.7)$$

$$R U \mapsto S(\text{hom}_R(X,Y)) \quad (2.8)$$

The crux is that $F$ and $G$ are adjoint functors (although I have no idea how to prove this). This means that given any $RY$ and $SZ$ we have a natural isomorphism

$$\text{hom}_R(FZ,Y) \cong \text{hom}_S(Z,GY) \quad (2.9)$$

More transparently:

$$\text{hom}_R(X \otimes Z,Y) \cong \text{hom}_S(Z,(\text{hom}_R(X,Y))) \quad (2.10)$$
3 The Induced representation

Let us now get to the point. If $H \leq G$ and $W$ is an $H$-representation, this means we have a homomorphism $\rho : H \to \mathbb{GL}(W)$. Equivalently, we can say we have a ring homomorphism from the group ring $\mathbb{C}H$ into the ring of endomorphisms of $W$, $\text{End}(W) : \rho : \mathbb{C}H \to \text{End} W$. This makes End $W$ a left $\mathbb{C}H$ module. $\mathbb{C}G$ is of course a left $\mathbb{C}G$-module, but it is also a right $\mathbb{C}H$ module, so we can take the tensor product $\mathbb{C}G \otimes W$. The resulting $\mathbb{C}G$-module, $\mathbb{C}G \otimes W$ (sometimes written Ind $W$) is the induced representation. This is analogous to the process of changing the field of a vector space (from $\mathbb{R}$ to $\mathbb{C}$ say) by tensoring by a field which contains the original field as a sub-field. Suppose $U$ is an arbitrary $G$ representation - that is, $U$ is a left $\mathbb{C}G$-module. Let us now apply (2.10):

$$\text{hom}_{\mathbb{C}G}(\mathbb{C}G \otimes W, U) \cong \text{hom}_{\mathbb{C}H}(W, \text{hom}_{\mathbb{C}G}(\mathbb{C}G, U))$$ (3.1)

We now use the fact that $\text{hom}_{\mathbb{C}G}(\mathbb{C}G, U) \cong U$ via the isomorphism $f \mapsto f(1)$ to conclude:

$$\text{hom}_{\mathbb{C}G}(\mathbb{C}G \otimes W, U) \cong \text{hom}_{\mathbb{C}H}(W, U)$$ (3.2)

which is a statement of Frobenius reciprocity provided we consider $U$ as an $H$-representation by restriction.

4 Minor holes

1. I am not entirely sure that I have the correct statement of the hom-tensor adjunction formula for left modules